Sensitivity analysis for models with dynamic inputs: a case study to control the heat consumption of a real passive house

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ABSTRACT: *In this communication, we perform the sensitivity analysis of a building energy model. The aim is to assess the impact of the weather data on the performance of a model of a passive house, in order to better control it. The weather data are uncertain dynamic inputs to the model. To evaluate their impact, the problem of generating coherent weather data arises. To solve it, we carry out the Karhunen-Loève decomposition of the uncertain dynamic inputs. We then propose an approach for the sensitivity analysis of this kind of models. The originality for sensitivity analysis purpose is to separate the random variable of the dynamic inputs, propagated to the model response, from the deterministic spatio/temporal function. This analysis highlights the role of the solar gain on a high-insulated passive building, during winter time.*

KEYWORDS: *Global Sensitivity Analysis; Karhunen-Loève Expansion; Dynamic Inputs; Building Energy Model*

1 INTRODUCTION

Nowadays, one important feature when designing high-performance buildings is to reduce the energy consumption, for economical but also for environmental purposes. Thus, it appears a need of strategy to better control and optimize the heat consumption of houses. In this framework, softwares for building performance simulation are developed and are constantly updated due to the evolution of materials, techniques, ... This leads to complex, high-dimensional and multi-physics models, presenting uncertain inputs due to measurements, expert judgements. Thus, the problem of their reliability arises. To assess it, it requires some tools to better understand the influence of the parameters responsible for the heat consumption in order to control it. Uncertainty and sensitivity analyses (UASA) can help answering it and evaluating the impact of this lack of knowledge onto the model responses [15, 7, 22]. Uncertainty analysis aims at quantifying the overall uncertainty within a model and sensitivity analysis aims at determining the most influent input parameters onto the model response. Numerous studies have focused on the sensitivity analysis for static non-linear models, i.e. presenting static inputs, for example [22, 1, 3, 5, 8, 24]. The approaches may be local or global. Local approaches help to determine the impact of a small input variation around a nominal value [28]. Global approaches also allow the determination of the same impact but by varying the input in its entire range of variation. For some applications, as energy building, it can be of great importance to consider the entire uncertainty range of inputs since they can vary within large intervals depending on their meaning. Considering only a single point in that interval as in the case of local study may be not enough informative and robust. Another advantage of global sensitivity analysis is that the sensitivity estimates of individual parameters are evaluated while all the other inputs are varied. In this way, the relative variability of each input is taken into account, thus revealing any existing interactions. Global methods are often based on the analysis of the output variance and are known as ANOVA (ANalysis Of VAriance) techniques [13, 21]. The present study exclusively focuses on global approaches.

Regarding building energy modeling, the literature focuses on sensitivity analysis for building models presenting only static inputs, for instance the thermo-physical properties of the materials [27]. However, some phenomena previously neglected for energy-consuming building may become preponderant in the consumption of low energy buildings, as the consideration of weather data, not only the temperature but also the solar radiation, the relative humidity, the wind speed,... But, it is expected that meteorological inputs play a crucial role in designing high-performance buildings. There are very few studies concerning the influence of the weather data which depend on time and are thus seen as dynamic uncertain inputs for the models. This can be explained by the fact that generating coherent weather data randomly is not obvious. One solution is to use experimental measure but it involves a large amount of data, too expensive to manage. Thus, an alternative is to create representative long-term weather data file [4]. But, while generating static inputs samples is not an issue, it is not straightforward to generate samples that satisfy the desired random fields distribution.

The aim of this study is then to assess the impact of the weather data on the heat consumption of a real passive house. This problem is challenging because it involves the one of the sensitivity analysis for models with uncertain dynamic inputs. Such an issue is rarely addressed (except, for instance, in [16]).

The uncertain static inputs can be seen as random variables defined by their marginal distribution and the dynamic ones as random fields, defined by their covariance function. The random variables can then be generated using a random sampling method, for instance the Latin Hypercube random sampling [12]. A random field can be represented as a series expansion involving a complete set of deterministic functions with corresponding random coefficients [23, 11, 29], as the truncated Karhunen-Loève (KL) expansion [14, 20]. KL series expansion is based on the eigen-decomposition of the covariance function, involving orthogonal deterministic basis functions and the orthogonal uncorrelated random coefficients. This allows the optimal encapsulation of the information contained in the random process into a set of discrete uncorrelated random variables.

The originality of the proposed approach is to separate the random variable of the dynamic inputs, propagated to the model response, from the deterministic spatio/temporal function, using Karhunen-Loève decomposition of the dynamic inputs. Then, for the SA, the influence of the dynamic inputs onto the model response is given by the one of the random coefficients of the Karhunen-Loève decomposition. For the estimation of their sensitivity indices, the Sobol' method is applied [24]. The proposed approach is then applied to a building energy model to quantify the impact of each weather data input on the performance of a real passive house.

The paper is organized as follows. In Section 2, the energy building model which is studied is described. Then, Section 3 is focused on the generation of coherent weather data. In Section 4, sensitivity analysis for static inputs is recalled and then is extended to dynamic inputs. Finally, Section 5 presents the results obtained.

2 ENERGY BUILDING MODEL

The building energy model studied in this work represents a real building located on an experimental platform named INCAS, introduced by the National Institute of Solar Energy and located in Le Bourget Du Lac, in France. This place presents a temperate continental climate with alpine influence. The building is a single-detached house with low energy consumption (figure 1). The house is double-glazed. The thickness of the wall is 50 cm, made of 15 cm of perpend, 20 cm of insulating material and again 15 cm of perpend. The house contains hundreds of sensors to quantify its thermal behavior. Details about the house can be found in [25]. The building is divided

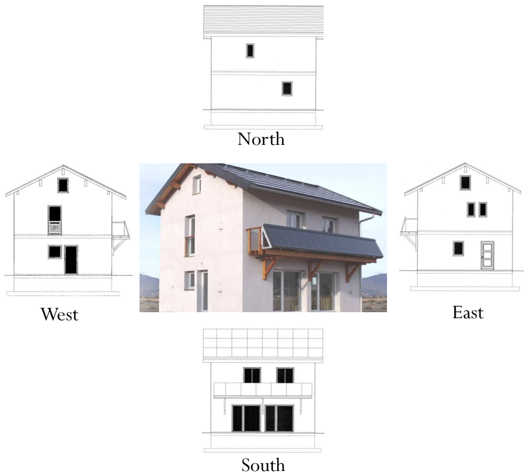

Figure 1: INCAS house

into several thermal zones, but only the ground floor and the first floor are studied. The solar gains are maximized in winter and minimized in summer thanks to the glazed surfaces distribution and solar shading. The goal of the study is to quantify and characterize the capacity of the building to exploit environmental energy gain.

The output of interest is the heating consumption at the ground and at the first floors. The uncertain dynamic inputs are the outdoor air temperature, the direct and the diffuse solar radiations, the speed and the direction of wind and the relative humidity.

A previous study [10], not presented here, has focused on the statistical analysis of the weather data. One month of january that is representative of typical winter from 20 years of observations has been used, since the heat consumption is the highest and the most costly, during this period of time, at this place. Moreover, considering only one season allows to respect stationarity of data.

The statistical characteristics of a given input are the covariance function and the hourly mean, on one day. The next section describes how to generate coherent weather data randomly.

3 WEATHER DATA GENERATION

The weather data are considered as the uncertain dynamic inputs of the model. Let denote them as $\omega^d(\theta,x) = \{\omega_1^d(\theta,x),\ldots,\omega_{N_d}^d(\theta,x)\}$. Here $N_d = 6$. The stochastic variable θ is used to indicate the randomness of the input ω^d. The variable x represents the time dependence of ω^d but for some other applications, it can also represent a space dependence. The random fields ω_i^d are assumed independent and normally distributed with mean $\bar{\omega}_i{}^d(x)$ and a covariance function $C_i(x_1,x_2)$, $i = 1,\ldots,N_d$, N_d denoting the number of dynamic inputs. The covariance function $C_i(x_1,x_2)$ is symmetric, positive definite.

According to [20], the random fields ω_i^d, having a mean $\bar{\omega}_i{}^d(x)$ and a finite variance σ_i^2, can be approximated using the truncated KL series:

$$\omega_i^d(x,\theta) \simeq \bar{\omega}_i{}^d(x) + \sum_{ki=1}^{M_i} \sqrt{\lambda}_{ki}\xi_{ki}(\theta) f_{ki}(x) \tag{1}$$

where λ_{ki} and f_{ki} are the deterministic eigenvalues and eigenfunctions of the covariance function $C_i(x_1,x_2)$, ξ_i is a set of independent standard normal variables and M_i is the number of KL-terms. In practice, we retain the first M_i eigenmodes that contain the 95% of the variance of the input ω_i^d. The number of eigenmodes retained depends on the choice of the covariance function and may be very different from one input to another.

The key feature for simulating random fields using KL expansion lies on the ability to determine accurately the eigenvalues and eigenfunctions of the covariance function. They are given from the spectral decomposition of the covariance function $C_i(x_1,x_2)$, requiring to solve the homogeneous Fredholm integral equation of the second kind given by:

$$\int_{\mathscr{D}} C_i(x_1,x_2) f_{ki}(x_1)\mathrm{d}x_1 = \lambda_{ki} f_{ki}(x_2) \tag{2}$$

For some covariance functions (first-order Markov or Wiener-Levy processes, for example), the equation (2) can be twice differentiable with respect to x_1. The resulting differential equation can be solved analytically and eigenvalues can be obtained as well. But in most cases, solving equation (2) requires numerical methods, such as the Galerkin one [9]. The Galerkin approach consists of finding a functional basis for the solution space of the equation, then projecting the solution on the functional basis, and minimizing the residual with respect to the functional basis. Very often, the bases are polynomial or trigonometric. Representation of integral operators using polynomial or trigonometric bases requires approximate integration quadratures. This can be very costly to compute. To avoid tedious quadratures and alleviating computational effort, an alternative is to use a wavelet-Galerkin approach [20, 17]. In this case, the representation of integral operators is made in wavelet basis and can be performed without numerical integration, as detailed in the Appendix. The comparison of wavelet-Galerkin method with other available methods in solving the Fredholm integral equation can be found in [14].

Once the weather data are generated, the sensitivity analysis can be performed as explained in the next section.

4 SENSITIVITY ANALYSIS

First, let recall the way of performing sensitivity analysis of a model presenting static inputs. To do so, consider the following model:

$$\mathbf{y}(\theta) = g(\omega^s(\theta)) \tag{3}$$

where $\mathbf{y} \in \mathbb{R}^m$ is the model response of interest, $\omega^s = \{\omega_1^s,\ldots,\omega_{N_s}^s\}$ is a set of N_s random variables (static inputs) and g a nonlinear function. In the following, we assume that random variables ω_i^s are independent and defined by their marginal distribution. They can be generated using a random sampling, Latin Hypercube sampling scheme, for instance, for its ease of implementation, [12]. The effect of the uncertain inputs ω_i^s onto y_j ($y_j \in \mathbb{R}$) can be estimated with sampling-based methods such as Sobol' method [24]. The first-order sensitivity index of the input ω_i^s to the model output y_j is given by:

$$S_i = \frac{V(E(y_j|\omega_i^s))}{V(y_j)} \tag{4}$$

where $V(E(y_j|\omega_i^s))$ is the variance of the conditional expectation of y_j when ω_i^s is set and $V(y_j)$ is the total variance of the output y_j. The first order sensitivity index S_i represents the main effect of the input ω_i^s which corresponds to its contribution alone. The value of S_i lies between 0 and 1. The closer to 1 its value is, the more input ω_i^s contributes to the total variance of the output. Sensitivity indices of higher order can also be computed to assess the influence of the input interactions but it is not presented here, see [22] for more details.

Some models may be complex with a high number of inputs so that analytical calculations of the previous sensitivity index become time consuming or even impossible. It is therefore necessary to estimate them. Some approaches have been proposed in the literature, based on Monte Carlo simulations or spectral expansion of the output [5, 2, 6, 8, 26]. The estimator considered in this study is provided by [16] and the procedure to compute it from few samples to reduce the computational cost is recalled below.

With the method originally proposed in [24], $N_s + 1$ samples are necessary to compute the main effects of the N_s inputs. To reduce the simulation cost, a method proposed in [18] has been used, requiring only two samples.

Consider a first sample of inputs with K points, denoted

ω^{s1}:

$$\omega^{s1} = \begin{pmatrix} \omega^{s1}_{11} & \cdots & \omega^{s1}_{1K} \\ \omega^{s1}_{21} & \cdots & \omega^{s1}_{2K} \\ \cdots & \cdots & \cdots \\ \omega^{s1}_{N_s1} & \cdots & \omega^{s1}_{N_sK} \end{pmatrix} \tag{5}$$

The model is evaluated with this first input samples, leading to the output $\mathbf{y^{(1)}}$ of dimension $m \times K$, m components of the output, each with K sample points:

$$\mathbf{y^{(1)}} = \begin{pmatrix} y^{(1)}_{11} & \cdots & y^{(1)}_{1K} \\ y^{(1)}_{21} & \cdots & y^{(1)}_{2K} \\ \cdots & \cdots & \cdots \\ y^{(1)}_{m1} & \cdots & y^{(1)}_{mK} \end{pmatrix} \tag{6}$$

Associate a random permutation function RP_i to each input ω^s_i. A second set, denoted ω^{s2}, is built from ω^{s1} by permuting its values.

$$\omega^{s2} = \begin{pmatrix} RP_1(\omega^{s1}_{11}) & \cdots & RP_1(\omega^{s1}_{1K}) \\ RP_2(\omega^{s1}_{21}) & \cdots & RP_2(\omega^{s1}_{2K}) \\ \cdots & \cdots & \cdots \\ RP_{N_s}(\omega^{s1}_{N_s1}) & \cdots & RP_{N_s}(\omega^{s1}_{N_sK}) \end{pmatrix} \tag{7}$$

Then, the model is evaluated with this second set of input samples, to obtain $\mathbf{y^{(2)}}$:

$$\mathbf{y^{(2)}} = \begin{pmatrix} y^{(2)}_{11} & \cdots & y^{(2)}_{1K} \\ y^{(2)}_{21} & \cdots & y^{(2)}_{2K} \\ \cdots & \cdots & \cdots \\ y^{(2)}_{m1} & \cdots & y^{(2)}_{mK} \end{pmatrix} \tag{8}$$

where $y^{(2)}_j = RP_i(y^{(1)}_j)$ and RP_i is the random permutation function associated to the input ω^s_i.
To compute the first order sensitivity index (4) corresponding to the effect of ω^s_i, the values of $\mathbf{y^{(1)}}$ are rearranged according to the corresponding permutation RP_i. In this way, it is as the output $\mathbf{y}$ has been obtained when all the input factors are varying except the one of interest ω^s_i.
Let define $\| y_j \|^2 = y_j . y_j$, where y_j is a K-dimensional vector and the symbol (.) denotes the scalar product of two vectors. The following result gives a way for computing the first-order sensitivity index to the output y_j.

$$\hat{S}_i = \frac{(y^{(1)}_j - f^{(1)}u).(RP^{-1}_i(y^{(2)}_j) - f^{(2)}u)}{\| y^{(1)}_j - f^{(1)}u \| \| RP^{-1}_i(y^{(2)}_j) - f^{(2)}u \|} \tag{9}$$

with

$$f^{(1)} = \frac{1}{K}\sum_{k=1}^{K} y^{(1)}_{jk} \text{ and } f^{(2)} = \frac{1}{K}\sum_{k=1}^{K} RP^{-1}_i(y^{(2)}_{jk}) \tag{10}$$

and u the vector of K components equal to 1.

In the case of model presenting dynamic inputs, the problem of determining this sensitivity index is much more complex. In this case, the key point is to generate coherent samples of the dynamic input satisfying the desired random fields distribution. As explained in Section 3, the random fields can be decomposed using the KL expansion (1). The influence of the dynamic inputs can then be analyzed through the one of the random coefficients of the KL expansion. The sensitivity analysis is therefore performed as for static inputs, as explained below.

Now consider the following model:

$$\mathbf{y}(x,\theta) = g(\omega^d(x,\theta), \omega^s(\theta), x) \tag{11}$$

where $\mathbf{y} \in \mathbb{R}^m$ is the model response of interest, $x \in \mathscr{D}$ is the spatial/time variable, $\omega^d = \{\omega^d_1, \ldots, \omega^d_{N_d}\}$ is a set of N_d random fields (dynamic inputs) and $\omega^s = \{\omega^s_1, \ldots, \omega^s_{N_s}\}$ is a set of N_s random variables (static inputs). As previously, we assume that random variables ω^s_i are independent and defined by their marginal distribution.
The first step is to generate samples for the dynamic inputs using the KL decomposition of ω^d, as detailed in Section 3. Once the eigenmodes are obtained for all the dynamic inputs $\omega^d_i(x,\theta)$ using (1), the influence of $\omega^d_i(x,\theta)$ is then given by the one of the random coefficients ξ_{ki} propagated to $\mathbf{y}(x,\theta)$. Consider the M_i-dimensional random coefficient ξ_i, grouping the M_i modes of the input ω^d_i:

$$\xi_i = \{\xi_{1i}, \ldots, \xi_{M_ii}\} \tag{12}$$

Thus, SA of the model output $\mathbf{y}(x,\theta)$ is performed through the random vectors $\{\xi_1, \ldots, \xi_{N_d}, \omega^s\}$, with ξ_i given by (12). Consequently, the effect of the group of factors ξ_i is the one of the dynamic input ω^d_i and so on. Thus, considering the static and the dynamic inputs of the model, there is $N_s + \sum_{i=1}^{N_d} M_i$ inputs in all to analyze.
The sensitivity indices of the group of factors $\xi_i = \{\xi_{1i}, \ldots, \xi_{M_ii}\}$ can be computed as:

$$S_i = \frac{V(E(y_j|\xi_{1i}, \ldots, \xi_{M_ii}))}{V(y_j)} \tag{13}$$

where $V(E(y_j|\xi_{1i}, \ldots, \xi_{M_ii}))$ is the variance of the conditional expectation of y_j when ξ_{1i}, $\xi_{2i}, \ldots$ and ξ_{M_ii} are set. This sensitivity index can be estimated as explained previously.
The sensitivity of ω^s can then be computed using (4) and the estimator (9), as presented previously.

The proposed approach is applied to the energy building model presented in Section 2.

5 RESULTS

Consider the energy building model of Section 2. The model output $\mathbf{y} = \{y_1, y_2\}$ is the ideal heating consump-

tion at the ground floor (y_1) and at the first floor (y_2). The daily consumptions are summed over one month, in such a way that the output $\mathbf{y}$ does no longer depend on time. However, the inputs ω^d are still depending on time. They represent the outdoor air temperature (ω_1^d), the direct and the diffuse solar radiations (ω_2^d and ω_3^d), the speed and the direction of wind (ω_4^d and ω_5^d) and the humidity (ω_6^d). There is no static inputs for this study.
From the hourly mean $\bar{\omega}_i^d$ and the covariance function $C_i(x_1,x_2)$, the sample generation of the dynamic inputs ω^d can be carried out, as explained in Section 3, using the Fast Haar wavelet algorithm (see the appendix). To do so, Matlab software has been used. This leads here to 512 modes for each of the 6 inputs, that is $M_i = 512$, $\forall i = \{1,\ldots,6\}$. Then, the model is simulated with the generated inputs, using the dedicated software Energy-Plus.

It can be noticed that the heat consumption is higher at the first floor than at the ground floor. The two thermal zones under study have the same volume but there is more glass surface at the ground floor. Thus, it may be assumed that the solar gain is more important at this floor, during winter.
The sensitivity indices are computed as explained in Section 4. The 512 ξ_{ij} of a given input are grouped together. Thus, there is 6 sensitivity indices S_i to compute, according to (13). To do so, the estimator (9) has been used. Moreover, we take the advantage of the random permutation trick explained in Section 4, which requires only two samples, here of $K = 1000$ points, allowing to reduce the computational cost. For this computation, Matlab software has been used.

Figure 2 shows the dispersion of the consumption.

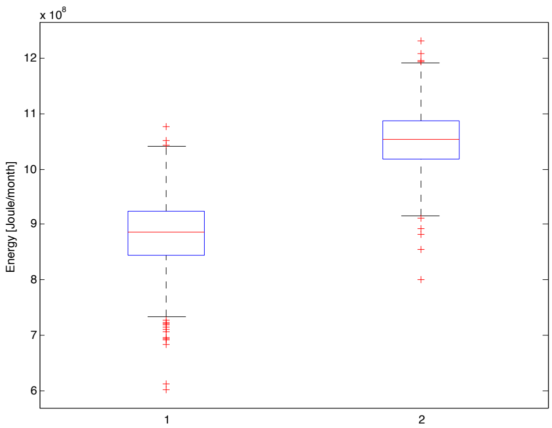

Figure 2: Dispersion of the heat consumption for the ground floor (1) and for the first floor (2)

Figure 3 represents the sensitivity indices for each input.

For the ground floor, the outdoor temperature and the solar radiation have almost the same sensitivity index

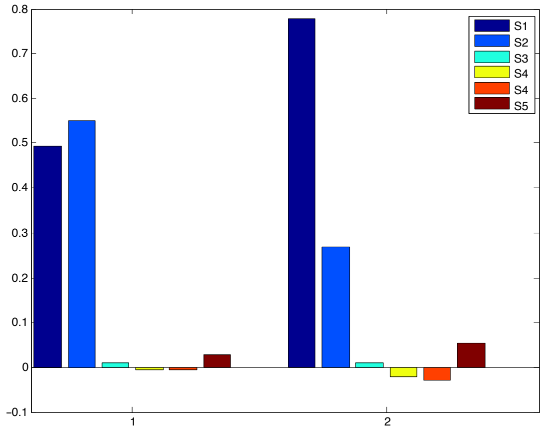

Figure 3: Sensitivity indices for the 6 inputs, at the ground floor (1) and at the first floor (2)

($S_1 = 0.46$ and $S_2 = 0.52$). This means that they influence the consumption almost in the same way, but with a slightly prevalence of the solar radiation. Besides, for the first floor, the outdoor temperature has a sensitivity index of $S_1 = 0.76$, explaining almost 80% of the consumption, compared to low influence of the solar radiation ($S_2 = 0.23$). On the other hand, the sensitivity indices of the humidity, diffuse radiation, velocity and direction of wind can be neglected showing that they are not influential. Note that small sensitivity indices can be negative because of the numerical approximation of the estimator considered. It is worth noting that these inputs are not influential in the studied model. This conclusion can be different for other models, in the way these variables are taken into account for the calculation of the energy balance. For example, humidity is not relevant here because the ideal heating device considered does not account for the impact of the humidity onto the consumption.
It can be noticed that the sum of the six sensitivity indices is almost one, showing no interaction between the inputs.
In conclusion, there is two significant weather inputs, the solar radiation and the outdoor temperature with different influences according to the given floor. These results highlight the capacity of the INCAS house to exploit the solar gains during winter, especially at the ground floor. Indeed, the solar radiation has a greater impact on a high-insulated passive building than on an energy-consuming one. This is relevant to minimize the heat consumption during winter in this low energy house.

6 CONCLUSION

In this paper, the impact of the weather data on the energy consumption of a passive house has been quantified. This has raised the problem of the generation of coherent weather data. The proposed solution is based on the KL decomposition of the dynamic inputs. An approach for sensitivity analysis of models presenting dynamic inputs

has been proposed. The originality is to separate the random coefficients of the input propagated to the output from the spatio/temporal function.
The study has shown that the outdoor temperature and the solar radiation are of prime importance for the INCAS house to reduce the heat consumption. It underlines the capacity of the INCAS house to exploit the solar gains during winter and to confirm its passive strategy to earn energy. This analysis helps to better understand the weather data influence for the heat consumption of a real passive house. It also helps to optimize the capacity of the building to exploit free energy resources. It is worth noting that the results depend on the building of interest (single-detached or terraced house, private or industrial building, ...), its orientation, the climate where it is located. SA can be seen here as a methodology to characterize the ability of low efficiency building to exploit free resources of the environment.
The next step is to study the influence of both static and dynamic inputs for the heat consumption, the static inputs representing the thermophysical properties of the materials.

Appendix

Using the wavelet-Galerkin approach, the covariance function $C_i(x_1,x_2)$ can be expanded as, in the wavelet bases:

$$C_i(x_1,x_2) = \sum_{k=0}^{N}\sum_{j=0}^{N} A_{kj}\psi_k(x_1)\psi_j(x_2) = \Psi^T(x_1)A\Psi(x_2) \tag{14}$$

where A is an $N\times N$ matrix representing the 2D wavelet transform of $C_i(x_1,x_2)$. The components A_{kj} of the matrix A are then given by:

$$A_{kj} = \frac{1}{h_k h_j}\int_0^1\int_0^1 C_i(x_1,x_2)\psi_k(x_2)\psi_j(x_1)\mathrm{d}x_1\mathrm{d}x_2 \tag{15}$$

where the functions ψ_k, $k = 1,\ldots,N$, are Daubechies' wavelets and h_k defined by:

$$h_k = \int_0^1 \psi_k^2(x)\mathrm{d}x \tag{16}$$

We consider the Haar wavelets, the simplest family of Daubechies' wavelets. The Haar mother wavelet is defined by:

$$\psi(x) = \begin{cases} 1 & \text{if} \quad x\in[0;0.5[\\ -1 & \text{if} \quad x\in[0.5;1[\\ 0 & \quad\text{otherwise} \end{cases} \tag{17}$$

A family of orthogonal Haar wavelets over the domain $[0;1]$ can be generated by shifting and scaling the mother wavelet:

$$\begin{aligned} &\psi_0(x) = 1 \\ &\psi_k(x) = 2^{a/2}\psi(2^a x - b), \qquad a,b\in\mathbb{Z} \\ &k = 2^a + b, \quad b = 0,1,\ldots,2^a-1, \quad a = 0,1,\ldots,m-1 \end{aligned} \tag{18}$$

where a and b are the dilation and translational integer constants, m the maximum wavelet level, related to N by $N = 2^m$.
The double integral of (15) corresponds to the 2D wavelet transform of $C_i(x_1,x_2)$ and does not require numerical integration. To carry out this 2D wavelet transform, the 1D wavelet transform is applied first on the rows and then on the columns of the matrix containing the values of $C_i(x_1,x_2)$ sampled over a N by N grid. The different steps to perform the 1D Haar wavelet transform are recalled below. For further details, see [19, 20].

1. Consider a function $F(x)$ (here the rows of $C_i(x_1,x_2)$).
   Sample $F(x)$ at N discrete points $x_i = \dfrac{2i+1}{2N}$, $i = 0,\ldots,N-1$. The samples are denoted F_i and $N = 2^m$ with m the maximum level of the wavelet, set a priori.
2. Initialize a $N\times 1$ vector denoted $a_{m,k}$ with the samples of F as follows:
   $$a_{m,k} = F_k, \qquad k = 0,\ldots,N-1 \tag{19}$$
3. The vector is processed using an inverse binary tree, where the topmost layer (mth) contains N nodes with values given by (19). The nodal values in subsequent layers are computed as:
   $$a_{j,k} = \frac{1}{2}(a_{j+1,2k} + a_{j+1,2k+1}) \tag{20}$$
   with $k = 0,1,\ldots,2^j-1$ and $j = m-1,\ldots,1,0$.
4. The wavelet coefficients are evaluated from the nodal values in this binary tree:
   $$c_{j,k} = \frac{1}{2}(a_{j+1,2k} - a_{j+1,2k+1}) \tag{21}$$
5. Finally, the wavelet coefficients are given by $[a_{0,0},c_1,\ldots,c_{N-1}]$, with:
   $$c_i = c_{j,k}, \quad i = 2^j + k \tag{22}$$
   where $k = 0,1,\ldots,2^j-1$ and $j = 0,1,\ldots,m-1$.
6. The function $F(x)$ can then be approximated as:
   $$F(x) = a_{0,0}\psi_0(x) + \sum_{i=1}^{N-1} c_i\psi_i(x) \tag{23}$$
   with $\psi_i(x)$ the Haar wavelets.

To solve (2), the eigenfunctions f_{ki} are approximated by Haar wavelet series:

$$f_{ki}(x) = \sum_{j=0}^{N-1} d_j^{(ki)}\psi_j(x) = \Psi^T(x)D^{(ki)} \tag{24}$$

where $D^{(ki)}$ a $N \times 1$ matrix whose components are the wavelet coefficients $d_j^{(ki)}$ and $\Psi^T(x)$ is the vector of components $\psi_i(x)$. Substituting equations (14) and (24) into equation (2) gives the following relation:

$$\lambda_{ki}\Psi^T(x)D^{(ki)} = \Psi^T(x)AHD^{(ki)} \tag{25}$$

where H is diagonal matrix of constants h_k due to the orthogonality condition. This leads to a finite dimensional eigenvalue problem of the form $\lambda_{ki}D^{(ki)} = AHD^{(ki)}$, which can be solved easily using an eigenvalue solver.

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